Last time : how to diagonalize a square matrix $B \in \mathbb{R}^{n \times n}$

find P invertible and D diagonal s.t. $B = PDP^{-1}$

Step 1 : find a basis $v_1, \dots, v_n$ of $\mathbb{R}^n$ so that

- B dilates v_1 by a factor of $\lambda_1 \iff Bv_1 = \lambda_1 v_1$

$\vdots$

- B dilates v_n by a factor of $\lambda_n \iff Bv_n = \lambda_n v_n$

$(\lambda_1, \dots, \lambda_n)$ are called eigenvalues of B

$(v_1, \dots, v_n)$ are called eigenvectors of B

Step 2 : set $P = \left(v_1 | \dots | v_n \right)$ and $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

Step 3 : profit! (i.e. with above choices, $B = PDP^{-1}$)

This procedure can be carried out for almost all matrices B, but at the cost of having complex eigenvalues/vectors

$\mathbb{C} = \{ z = a + bi \mid a, b \in \mathbb{R} \}$ where $i^2 = -1$

- complex numbers can be added, subtracted, multiplied, divided
- $z = a+bi \rightsquigarrow$ conjugate $\bar{z} = a - bi$
- absolute value $|z| = \sqrt{a^2+b^2} \rightsquigarrow |z|^2 = z \cdot \bar{z}$
- $\overline{(\bar{z})} = z$, $\overline{zw} = \bar{z}\,\bar{w}$, $\overline{\left(\frac{1}{z}\right)} = \frac{1}{\bar{z}}$

$$z = a + bi = r(\cos\theta + i \sin\theta)$$

Cartesian coordinates

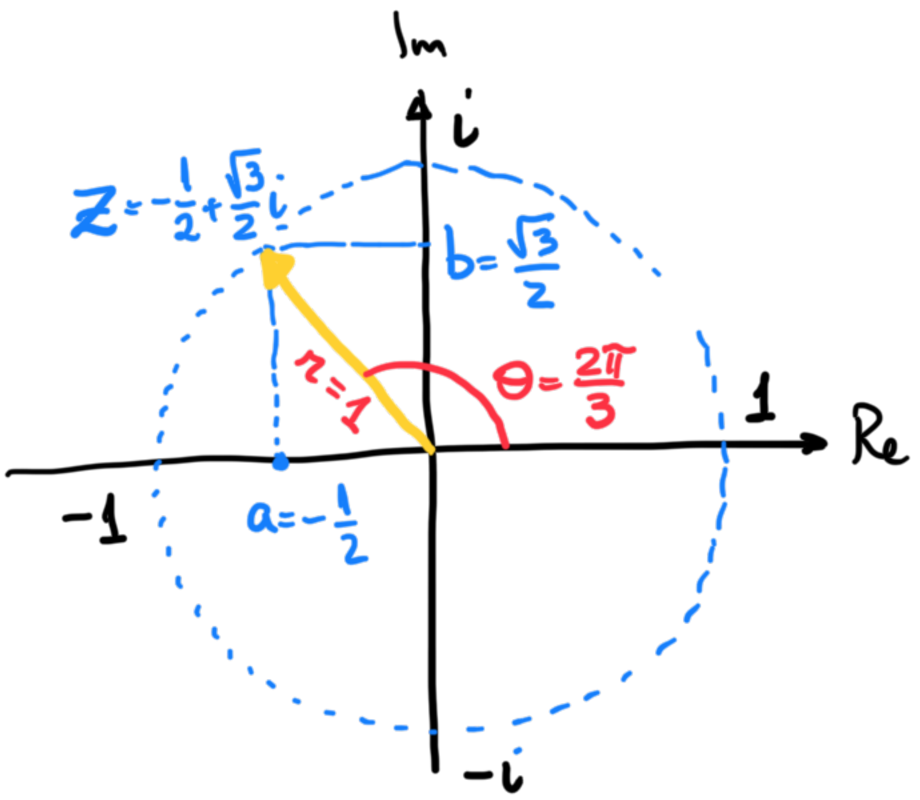
$a = \operatorname{Re} z$, $b = \operatorname{Im} z$

polar coordinates

$r = |z|$, $\theta = \arg(z) \in [0, 2\pi)$ modulo 2π

$r = \sqrt{a^2+b^2}$, $\theta = \arctan\frac{b}{a}$

- $|zw| = |z||w|$
- $\arg(zw) = \arg(z) + \arg(w)$
- $\left|\frac{1}{z}\right| = \frac{1}{|z|}$
- $\arg\left(\frac{1}{z}\right) = -\arg(z)$



Today: let's do computations with complex numbers in

Cartesian coordinates | polar coordinates

$$z = -\frac{1}{2} + \frac{\sqrt{3}}{2} i \quad = \quad z = 1\left(\cos\frac{2\pi}{3} + i \cdot \sin\frac{2\pi}{3}\right)$$

$$z^3 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^3$$

$$= \left(-\frac{1}{2}\right)^3 + \left(\frac{\sqrt{3}}{2} i\right)^3 + 3\left(-\frac{1}{2}\right)^2 \frac{\sqrt{3}}{2} i + 3\left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2} i\right)^2$$

$$= -\frac{1}{8} + \cancel{\frac{3\sqrt{3}}{8}} \underbrace{i^3}_{= -i} + \cancel{\frac{3\sqrt{3}}{8}} i - \frac{3 \cdot 3}{8} \underbrace{i^2}_{= -1} = -\frac{1}{8} + \frac{9}{8} = 1$$

$$\Downarrow$$

$$z^3 = 1$$

$$|z^3| = |z|^3 = 1$$

$$\arg(z^3) = 3 \cdot \arg(z) = 2\pi = 0$$

$$\Downarrow$$

$$z^3 = 1(\underbrace{\cos 0}_{= 1} + i \cdot \underbrace{\sin 0}_{= 0}) = 1$$

Today: complex numbers allow us to solve polynomial equations

e.g. $t^3 - 1 = 0$:

in $\mathbb{R}$, only solution is $\{1\}$

in $\mathbb{C}$, three solutions $\left\{1, -\frac{1}{2} + \frac{\sqrt{3}}{2} i, -\frac{1}{2} - \frac{\sqrt{3}}{2} i\right\}$

Solving a polynomial equation

$\Updownarrow$

Factoring the polynomial in linear terms

$$(t^3-1) = (t-1)(t^2+t+1)$$

over $\mathbb{R}$, no solutions

over $\mathbb{C}$, solutions $-\frac{1}{2}+\frac{\sqrt{3}}{2}i\ ,\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i$

$$(t^2+t+1) = \left(t-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)\left(t-\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right)$$

$$\Downarrow$$

$$t^3-1 = (t-1)\left(t-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)\left(t-\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right)$$

Quadratic formula: $at^2+bt+c=0$

assume real, but formula also works for complex

real roots:

- $t_1 = -\frac{b}{2a} + \frac{\sqrt{b^2-4ac}}{2a}\ ,\ t_2 = \frac{-b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}$ if $b^2-4ac>0$
- $t_1 = t_2 = -\frac{b}{2a}$ if $b^2-4ac=0$

complex roots:

- $t_1 = -\frac{b}{2a} + \frac{i\sqrt{4ac-b^2}}{2a}$, $t_2 = -\frac{b}{2a} - \frac{i\sqrt{4ac-b^2}}{2a}$ if $b^2-4ac<0$

$$at^2+bt+c = a(t-t_1)(t-t_2)$$

need to put a in RHS to match coefficient of t^2

THM 18.1: any degree n polynomial $P(t)$ has a complete set of $\underbrace{n \text{ roots}}_{\lambda_1,\dots,\lambda_n,\ \text{may have } \lambda_i=\lambda_j}$ in $\mathbb{C}$

$\Updownarrow$

$$P(t) = \text{const}\,(t-\lambda_1)(t-\lambda_2)\dots\dots(t-\lambda_n)$$

Ex: $t^3-6t^2+12t-8 = (t-2)(t-2)(t-2)$

root 2 appears with multiplicity 3

$$t^4-1 = (t^2-1)(t^2+1)$$
$$= (t-1)(t+1)(t^2+1)$$

$\lambda_1 = 1$ $\quad\lambda_2 = -1$

$t^2+1=0$ in $\mathbb{C}$?

$\Updownarrow$

$= (t-1)(t+1)(t-i)(t+i)$

$\lambda_3 = i \qquad \lambda_4 = -i$

$t = -1$ in $\mathbb{C}$

$\Updownarrow$

$t = i, \ t = -i$

DEF 18.2: let $B \in \mathbb{R}^{n \times n}$; we call

$\lambda \in \mathbb{C}$ an *eigenvalue* of B

$v \in \mathbb{C}^n \setminus 0$ an *eigenvector* of B

if $\boxed{Bv = \lambda v}$

A matrix $B \in \mathbb{R}^{n \times n}$ can have $1, 2, \dots, n$ eigenvalues

A matrix $B \in \mathbb{R}^{n \times n}$ can have $1, 2, \dots, n$ linearly independent eigenvectors

THM 18.3: An $n \times n$ matrix is diagonalizable

$\iff$ it has n linearly independent eigenvectors

Once you know that λ is an eigenvalue of B

it is easy to find an eigenvector for λ

$\Downarrow$

$$(B-\lambda I_n)v=0 \iff Bv-\lambda v=0 \iff Bv=\lambda v$$

once B,λ are known, solving for v is just a homogeneous equation

How to find the eigenvalues $\lambda_1,\dots,\lambda_n$ of B?

Proof of Thm 18.3 : $v_1,\dots\dots,v_n$ linearly independent

$$\lambda_1,\dots\dots\lambda_n \in \mathbb{C}$$

$$Bv_1=\lambda_1 v_1 \quad \dots\dots\dots\dots \quad Bv_n=\lambda_n v_n$$

$\underline{v}=(v_1,\dots,v_n)$

$\underline{e}=(e_1,\dots,e_n)$

$$P_{\underline{e}\leftarrow\underline{v}}=P=\begin{pmatrix} v_1 & \dots & v_n \end{pmatrix}$$

$$P^{-1}BP=P_{\underline{v}\leftarrow\underline{e}}\,B\,P_{\underline{e}\leftarrow\underline{v}}=\text{"}B \text{ in the } \underline{v} \text{ basis"}$$

$$\|$$

$$P^{-1}BP = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

$$\|$$

$$B = P\begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{pmatrix} P^{-1}$$

$\square$

How to effectively calculate eigenvalues?

$$\det(B) = \cancel{\det(P)} \det\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \underbrace{\cancel{\det(P^{-1})}}_{\det(P)^{-1}} = \lambda_1 \lambda_2 \cdots \lambda_n$$

Fact: product of eigenvalues $= \det(B)$

Let's generalize the idea above ; start from

$$B = P\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} P^{-1}$$

subtract $\underbrace{t}_{\text{any scalar}} \cdot I_n$

$$B - t\cdot I_n = P\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} P^{-1} - t\cdot P I_n P^{-1}$$

$$B - t\cdot I_n = P\left[\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ \end{pmatrix} - t\cdot I_n\right] P^{-1}$$

$$B - t\cdot I_n = P \begin{pmatrix} \lambda_1 - t & & & 0 \\ & \lambda_2 - t & & \\ & & \ddots & \\ 0 & & & \lambda_n - t \end{pmatrix} P^{-1}$$

multiply by -1

$$\boxed{t\cdot I_n - B = P \begin{pmatrix} t-\lambda_1 & & 0 \\ & \ddots & \\ 0 & & t-\lambda_n \end{pmatrix} P^{-1}}$$

$\Downarrow$

$$\det(t\cdot I_n - B) = \cancel{\det(P)}\, \det\begin{pmatrix} t-\lambda_1 & & 0 \\ & \ddots & \\ 0 & & t-\lambda_n \end{pmatrix} \cancel{\det(P^{-1})}$$

$$= (t-\lambda_1) \cdots (t-\lambda_n) \quad *$$

DEF 18.4: given an $n\times n$ matrix B, its **characteristic polynomial** is the degree n poly

$$\boxed{\chi_B(t) = \det(t\cdot I_n - B)}$$

COR 18.5 (follows from $\circledast$)

The eigenvalues of B are the roots of $\chi_B(t)$

$$\boxed{\chi_B(t) = (t-\lambda_1)\cdots\cdots(t-\lambda_n)}$$

Ex: find eigenvalues of $B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$

Step 1: calculate characteristic polynomial

$$\chi_B(t) = \det(t \cdot I_2 - B) = \det\left(t \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \right)$$

$$\chi_B(t) = \det\left(\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \right)$$

$$\chi_B(t) = \det\begin{pmatrix} t-2 & -1 \\ -1 & t-3 \end{pmatrix}$$

$$\chi_B(t) = (t-2)(t-3) - (-1)(-1)$$

$$\chi_B(t) = t^2 - 5t + 5$$

Step 2: the eigenvalues of B are roots of $\chi_B(t)$

these can be found by quadratic formula $\Longrightarrow$
$$\begin{cases} \lambda_1 = \frac{5}{2} + \frac{\sqrt{25-4\cdot5\cdot1}}{2} = \frac{5+\sqrt{5}}{2} \\ \lambda_2 = \frac{5}{2} - \frac{\sqrt{25-4\cdot5\cdot1}}{2} = \frac{5-\sqrt{5}}{2} \end{cases}$$

Step 3: find eigenvectors v_1, v_2 for λ_1, λ_2, i.e.

$$B v_1 = \lambda_1 v_1 \iff (B-\lambda_1 I_2)v_1 = 0 \iff \begin{pmatrix} 2-\frac{5+\sqrt5}{2} & 1 \\ 1 & 3-\frac{5+\sqrt5}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{pmatrix} 2-\frac{5+\sqrt5}{2} & 1 \\ 1 & 3-\frac{5+\sqrt5}{2} \end{pmatrix} = \begin{pmatrix} \frac{-1+\sqrt5}{2} & 1 \\ 1 & \frac{1+\sqrt5}{2} \end{pmatrix} \xrightarrow{\text{swap rows}} \begin{pmatrix} 1 & \frac{1+\sqrt5}{2} \\ \frac{-1+\sqrt5}{2} & 1 \end{pmatrix}$$

subtract $\frac{-1+\sqrt5}{2}$ times row 1 from row 2

$$\begin{pmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ 0 & 1-\frac{1+\sqrt{5}}{2}\cdot\frac{-1+\sqrt{5}}{2}=0 \end{pmatrix}$$

So $v_1 = \begin{pmatrix} X \\ Y \end{pmatrix}$ where $X + Y\cdot\frac{1+\sqrt{5}}{2} = 0$

$\Longrightarrow v_1 = Y\begin{pmatrix} -\frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}$ is an eigenvector for λ_1, $\forall Y \neq 0$

(any multiple of an eigenvector is an eigenvector)

Similarly, $v_2 = Y\begin{pmatrix} -\frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}$ is an eigenvector for λ_2

Ex: $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\rightsquigarrow$ eigenvalues? eigenvectors?

$$\chi_B(t) = \det(t\cdot I_2 - B) = \det\begin{pmatrix} t & 1 \\ -1 & t \end{pmatrix} = t^2+1$$

complex roots are $t = i$

eigenvalues $\lambda_1 = i$ $\Leftarrow$ $t=-i$

$\lambda_2 = -i$ $\qquad t^2+1 = (t-i)(t+i)$

Let's look for an eigenvector v_1 for $\lambda_1 = i$

Let's look for an eigenvector v_2 for $\lambda_2 = -i$

$$Bv_1 = \lambda_1 v_1 \iff (B - \lambda_1 I_2)\, v_1 = 0$$

$$\iff \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -i \\ -i+i & -1+i\cdot(-i) \end{pmatrix} = \begin{pmatrix} 1 & -i \\ 0 & -1-i^2=0 \end{pmatrix}$$

So eigenvectors $v_1 = \begin{pmatrix} X \\ Y \end{pmatrix}$ must satisfy $X - iY = 0$

$\Rightarrow$ an eigenvector for $\lambda_1 = i$ is $v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

Sanity check:

$$Bv_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ i \end{pmatrix} = \begin{pmatrix} i^2 \\ i \end{pmatrix} = i \begin{pmatrix} i \\ 1 \end{pmatrix} = \lambda_1 v_1$$

Similarly, an eigenvector for $\lambda_2 = -i$ is $v_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

$$\Downarrow$$

Diagonalization of $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is

$$B = P D P^{-1}$$

where $P = (v_1 \ \ v_2) = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$

$$D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Sanity check:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}^{-1}$$

90° rotation

Geometrically, we showed that rotation by 90° dilates the vector $\begin{pmatrix} i \\ 1 \end{pmatrix}$ by i

and the vector $\begin{pmatrix} -i \\ 1 \end{pmatrix}$ by $-i$

No wonder we can't see these directions which are dilated by rotation; they are complex, not real.